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On forms of integral and integro-differential equations of vibratory antennas and convergence of algorithms

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Abstract: *For dipole (vibratory) antennas we consider the forms of integral equations, their transformations and how it effects on convergence. The optically pumped carbon nanotubes were studied numerically.*

Keywords: *vibratory antenna, integral equations, CNT.*

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О формах интегральных и интегро-дифференциальных уравнений вибраторных антенн и сходимости алгоритмов их расчета

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Аннотация: *Для дипольных (вибраторных) антенн рассмотрены формы интегральных уравнений, их преобразования и влияние на сходимость. Численно исследованы углеродные нанотрубки с оптической накачкой.*

Ключевые слова: *вибраторная антенна, интегральные уравнения, углеродные нанотрубки.*

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1. Introduction

In 1888 H. Hertz have obtained the solution to the problem of a point electric dipole, and a decade later (in 1897) H. Pocklington published an article on the integral equation (IE) of a thin electric linear vibrator (LV) [1]. From this moment the countdown in the development of antenna theory and technology began. Further, other forms of IEs for the vibratory antenna (VA) were obtained by E. Hallen [2], then M. Leontovich and M. Levin [3], then L. Vainshtein,

P. Kapitsa and V. Fok [4, 5], L. Vainshtein [6–10], L. Vainshtein and V. Fok [11]. In a number of recent works [12–18] new modifications of the Pocklington IE are proposed, including those having the form of Fredholm IEs of the second kind, as well as those recorded with respect to the magnetic field [18]. These IEs have different forms than the original Pocklington equation (PE). The PE itself can be considered first as 1st-order Fredholm IE with an exact singular kernel, and second as a similar IE with an approximate continuous kernel. Thus, all transformations of the PE of thin LV boil down to how the integral operator can be transformed. For a number of years, the interest has been aroused by strip, microstrip and other planar antennas [19], which are similar in principle to wire VAs. Recently, there has been interest in THz, IR and even optical band antennas, structurally made in the form of carbon nanotubes (CNTs), graphene strips and other nanostructures. CNT and graphene have surface conductivity σ and can even be active (under optical pumping) [20]. They are promising for the transmission of THz and IR radiation at short distances. The final complex impedance requires an impedance approach to the formulation of the problem and can significantly change the properties of such antennas due to the occurrence of plasmons and other effects [21]. In a strict solution, all forms of equations (with the same formulation of the problem) should give the same results, but have different convergence and require different computational resources and methods of solution, which is the subject of this work. It is interesting to trace the methods of transformation of IEs and their impact on obtaining solutions. In addition, the paper considers Vas in the form of CNT and graphene strips, which are of interest for the transmission of THz and IR energy in microstructures and over short distances. Since the IE in the simplest architecture is a one-dimensional (1D), the methods of its transformation are clearer and simpler than for 2D and 3D IEs, which is why its consideration in the work is due. Often IE is understood not as an integral one with a smooth kernel, but as a singular IE (SIE). In this paper, modifications of IE and integrodifferential equations (IDEs), as well as one-dimensional SIE and singular IDE (SIDE) for an electrically thin and thick LV are obtained from PE of even from volume IE. Their connection with the PE (1897) and the Hallen equation (1938) is shown. LAs with impedance CNT and graphene nanoribbon are considered (see for compare [21–30]).

2. Problem statement

LV Fig. 1 since Pocklington, who based his work on the Hertz approach [31], is a classical object of IE theory. Usually the problem is considered in a simplified formulation: the cylinder is considered complete and perfectly conductive without end surfaces, and its excitation is carried out by a point voltage source through an infinitesimal gap. As seen in Fig. 1, in the technical implementation, it also includes two conductive cylinders 1 of radius a, usually powered from a shielded source 2 by a feeder 3. The feeder does not radiate, because the two wires of the last close, and the currents in them are antiphase. Therefore, only the currents flowing over the surface of the cylinders and the gap are radiating. The model for the gap has a very strong influence on the calculation of the input impedance, as well as the ends in the case of a thick VA. Usually the cylinders are considered perfectly conductive, and the current is superficial. Recently, there have been works taking into account the finite surface impedance, for example, [21–29]. Its accounting is important, for example, for the analysis of terahertz antennas based on CNT. For such bands, metal nanowires (quantum threads) can be used as LV. Their conductivity is quantized and the field penetrates deep, requiring the use of volume integral equations. Usually for theoretical research consider the simplest symmetrical LV. Asymmetry is not particularly difficult. Consideration of a thin VA in the form of a curved wire is convenient on the basis of the Hallen equation and leads to complication of the kernel of the integral operator. Another approximation is a hollow LV. In this case, the equations are quite simplified, since the end currents are not taken into account (Fig. 1). Another strong approximation is a thin VA. Usually the approximation $a/\lambda << 1$ is used: the smallness of the radius in relation to the wavelength. However, the ratio a/l characterizing the elongation of the cylinder is also important. The strongest approximation associated with the fineness of the VA is the azimuthal independence of the current density, i.e. the one-dimensionality of the IE. And although a number of publications have obtained results suitable for $a/\lambda \sim 1$, the condition $a/l \sim 1$ requires taking into account many azimuthal harmonics. It is connected, in particular, with asymmetry of excitation by feeder and external field (incident plane wave), with neighboring VA, etc. With symmetric feeding of a single VA without incident field, the density of the longitudinal current $J_z(\rho,z)=\eta_z(z)\delta(a-\rho)$ on the side surface depends only on the longitudinal coordinate *z*, and at the ends there is only a radial component $J_\rho(\rho,z)$. At the ends $J_\rho(\rho,z)=\eta_{\pm\rho}(\rho)\delta(z\mp\Delta/2)$ and $J_\rho(\rho,z)=\eta^{\pm}{}_{\rho}(\rho)\delta(z\mp(l+\Delta/2))$. Here $\eta_z(z)$, $\eta_{\pm\rho}(\rho)$ and $\eta^{\pm}{}_{\rho}(\rho)$ is the surface current density. Another significant approximation is the narrow gap $\Delta << l$. It is due to the fact that usually the current density $J_z = I_0\delta(z)\delta(\rho)$ in the gap is replaced by, which means continuity J_z. On the other hand, an electric field is excited in the gap $E_z \approx U_0\delta(z)$, which is

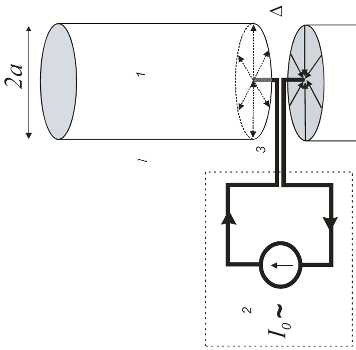

Fig. 1. Schematic view of LV: cylinders with ends 1, generator, feeder 2. Arrows show currents.

Рис. 1. Схематическое изображение линейного вибратора: цилиндры с концами 1, генератор, фидер 2. Стрелками показаны токи

proportional to the bulk density of the displacement electric current $J_z = i\omega\varepsilon_0 U_0 \delta(z)$. For a thin VA with a small gap, the conduction current passes into the bias current, so one-dimensional IE can be used in the region $-l < z < l$. For a wide gap, this is not possible: the bias current extends beyond the cylinder $-l \le z \le l$, $\rho \le a$, so a strict approach should include excitation by surface side and end currents. In addition, it is necessary to take into account two linear currents suitable for the points of power supply. They can be taken on the axes of these wires. Boundary conditions must be imposed on the entire surface of both cylinders and on the side surface of the supply wires. This is a strict statement of the problem. It allows you to determine the component E_z on the surface of the gap and calculate the average voltage in the gap $\overline{U}$. Then the input impedance of the gap is $Z_{in} = \overline{U} / I_0$. Since the diameter of the feeder wire $r_0 \le a$ is small, in this case the approximation of point feeding is quite justified: $J_z(\rho, z) = I_0 \delta(r_0 - \rho)\delta(z)$.

The LV from CNT corresponds to this consideration. Given the problem of excitation of such a VA, let consider a single radiating CNT excited in the optical range by laser pumping. The conductivity of CNTs is considered in [31, 33] and for big radius a limits to the graphene one for zigzag and armchair configurations. So we use the graphene conductivity, which has a negative real part due to the pumping power. In balance, the radiation power coincides with the pumping power, so the calculation of such antennas is very simple, especially for CNTs without ends: the power developed on the surface must coincide with the power emitted through the remote sphere Σ:

$$P = -\mathrm{Re}\oint_S \boldsymbol{\eta}(\mathbf{r})\mathbf{E}^*(\mathbf{r})\,ds/2 = -\mathrm{Re}\oint_S \sigma\left|\mathbf{E}(\mathbf{r})\right|^2 ds/2 = \mathrm{Re}\oint_\Sigma \mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\,ds/2 .$$

At the same time, the density of the surface current $\boldsymbol{\eta}$ satisfies IE. In general case of field penetration into the volume of the VA the IE

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{in}(\mathbf{r}) + Z_0\frac{k_0^2+\nabla\nabla\cdot}{ik_0}\int_V G(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')d^3r' . \quad (1)$$

For magnetic field we have

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_{in}(\mathbf{r}) + \nabla\times\int_V G(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')d^3r' . \quad (2)$$

These 3D volumetric IEs are suitable for finding the volumetric current density in the cylinder [34]. They should be used when the radius a is of the order of the skin layer δ. This is the case for nanowires, or for very high frequencies when the dielectric constant of the metal is almost real positive and the normal skin effect is not performed. Also it is the case for ultra-low frequencies when the penetration depth is great. When $\delta << a$ one can use the surface IE:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{in}(\mathbf{r}) + Z_0\frac{k_0^2+\nabla\nabla\cdot}{ik_0}\int_S G(\mathbf{r}-\mathbf{r}'_\tau)\boldsymbol{\eta}(\mathbf{r}'_\tau)d^2r' + \mathbf{z}_0E_{0z}(\mathbf{r}) + \boldsymbol{\rho}_0E_{0\rho}(\mathbf{r}), \quad (3)$$

$$E_{0z}(\mathbf{r}) = 2\pi I_0Z_0r_0\frac{G_{0zz}(\rho,z\,|\,r_0,0) + k_0^2G_0(\rho,z\,|\,r_0,0)}{ik_0},$$

$$E_{0\rho}(\mathbf{r}) = 2\pi I_0Z_0r_0\frac{G_{0z\rho}(\rho,z\,|\,\rho,0)}{ik_0}. \quad (4)$$

If $\mathbf{r}\in S$ this is 2D surface IE. Here $\mathbf{E}_{in}$, $\mathbf{H}_{in}$ are initial field, $Z_0=\sqrt{\mu_0/\varepsilon_0}$ is the impedance, G is the scalar Green's function (GF), G_0 is its the azimuth independent part, the subscripts correspond to the coordinate derivatives. Note that on the surface $\boldsymbol{\eta} = \sigma\mathbf{v}\times\mathbf{E}\times\mathbf{v} = \sigma\mathbf{E}_\tau$, so taking into account the finite surface impedance the IE (3) becomes the second kind IE. It is convenient to use 1D IE instead 2D one, using smallness of a. We have

$$G_0(\rho,z\,|\,\rho',z') = \frac{1}{4\pi}\int_0^\infty \frac{\exp\left(-\sqrt{\chi^2-k_0^2}\,|z-z'|\right)I_0(\chi\rho)K_0(\chi\rho')}{\sqrt{\chi^2-k_0^2}}\chi d\chi . \quad (5)$$

We will use multiple views of the GF $G=(4\pi R)^{-1}\exp(-ik_0R)$,

$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta)}$, $\theta = \varphi - \varphi'$, in particular

$$G = \sum_{n=0}^{\infty} [G_n \cos(n\theta) - iF_n \sin(n\theta)]. \tag{6}$$

The azimuth independent part also has the views

$$G_0(\rho, z \mid \rho', z') = \frac{1}{4\pi} \int_0^{\infty} \cos(-ih(z-z')) f_0(\kappa(h), \rho, \rho') dh, \tag{7}$$

$$G_0(\rho, z \mid \rho', z') = \frac{1}{8\pi} \int_{-\infty}^{\infty} \exp(-h(\kappa)|z-z'|) g_0(h(\kappa), \rho, \rho') h^{-1}(\kappa) \kappa d\kappa. \tag{8}$$

Here $h = \sqrt{\kappa^2 - k_0^2}$, $\kappa = \sqrt{h^2 - k_0^2}$, and the functions have the following form below. If $\rho < \rho'$, then $f_0(\kappa(h), \rho, \rho') = K_0(\kappa\rho') I_0(\kappa\rho)$, $g_0(h(\kappa), \rho, \rho') = K_0(h\rho') I_0(h\rho)$. If $\rho > \rho'$, then $f_0(\kappa(h), \rho, \rho') = I_0(\kappa\rho') K_0(\kappa\rho)$, $g_0(h(\kappa), \rho, \rho') = I_0(h\rho') K_0(h\rho)$. $I_0(z)$ and $K_0(z)$ mean McDonald's cylindrical functions of the second kind of complex argument. We have

$$G(\rho, \phi, z \mid \rho', \phi', z) = \frac{4\pi^2}{i\sqrt{\rho\rho'}} \sum_{m=0}^{\infty} (m+1/2) H_{m+1/2}^{(2)}(k_0\rho) J_{m+1/2}(k_0\rho') P_m(\cos(\theta)).$$

Here $\rho > \rho'$. Since there are also relations $R = \sqrt{u_+^2 + u_-^2 - 2u_+u_-\cos(\theta)}$, $2u_{\pm} = \sqrt{(z-z')^2 + (\rho+\rho')^2} \pm \sqrt{(z-z')^2 + (\rho-\rho')^2}$, then [4]

$$G(\mathbf{r} - \mathbf{r}') = \frac{4\pi^2}{i\sqrt{u_+u_-}} \sum_{m=0}^{\infty} (m+1/2) H_{m+1/2}^{(2)}(k_0 u_+) I_{m+1/2}(k_0 u_-) P_m(\cos(\theta)). \tag{9}$$

This allows to compute functions in the decomposition (6):

$$G_n = \frac{1}{\pi\varepsilon_n} \int_0^{2\pi} G \cos(n\theta) d\theta, \quad F_n = \frac{i}{\pi} \int_0^{2\pi} G \sin(n\theta) d\theta. \tag{10}$$

Here $\varepsilon_n = 1 + \delta_{n0}$ is the Neumann multiplier. Expressions (10) are given by series (9), in which instead of Legendre polynomials one should substitute the integrals from them I_{mn}^c and I_{mn}^s:

$$I_{mn}^{c}=\frac{1}{\pi}\int_{0}^{2\pi}P_m\left(\cos(\theta)\right)\cos(n\theta)\,d\theta=\frac{1+(-1)^{n+m}}{\pi}\int_{1}^{-1}\frac{P_m(x)T_n(x)\,dx}{\sqrt{1-x^2}},\quad(11)$$

$$I_{mn}^{s}=\frac{i}{\pi}\int_{0}^{2\pi}P_m\left(\cos(\theta)\right)\sin(n\theta)\,d\theta=i\frac{1+(-1)^{n+m}}{\pi}\int_{-1}^{1}P_m(x)U_{n-1}(x)\,dx\,.\quad(12)$$

In particular, $I_{m0}^{s}=0$, $I_{mn}^{c}=I_{mn}^{s}=0$, if $m+n$ id odd, $I_{(2m)0}^{c}=2\left[\Gamma^2\left(1/2+m\right)/m!\right]^2/\pi$, so we get another representation

$$G_0(\mathbf{r}-\mathbf{r}')=\frac{4\pi}{i\sqrt{u_+u_-}}\sum_{m=0}^{\infty}(m+1)H_{2m+1/2}^{(2)}(k_0u_+)I_{2m+1/2}(k_0u_-)\left[\frac{\Gamma(1/2+m)}{m!}\right]^2.(13)$$

For integral (12) one can fined

$$I_{mn}^{s}=2i\frac{\left(1+(-1)^{n+m}\right)(m-n+1)(m-n+3)...(m+n+1)}{\pi(m-n)(m-n+2)...(m+n)}.\quad(14)$$

We express the Legendre polynomials $P_n(x)$ using Chebyshev polynomials: $P_0=T_0$, $P_1=T_1$, $P_2=3/4T_2+1/4T_0$, $P_3=5/8T_3+3/8T_1$, $P_4=35/64T_4+5/16T_2+9/64T_0$, $\ldots$. To calculate subsequent expansions, we have obtained the recurrent formulas for the decomposition coefficients, but they are some cumbersome. It is convenient to calculate integrals using the orthogonality of Chebyshev polynomials: $I_{1n}^{c}=\delta_{n1}$; $I_{0(2n)}^{c}=2\delta_{n0}$; $I_{mn}^{c}=0$, if $n>m$ and if $n+m$ is odd. We have: $I_{22}^{c}=3/4$, $I_{31}^{c}=3/8$; $I_{33}^{c}=5/8$; $I_{40}^{c}=9/64$; $I_{42}^{c}=5/16$; $I_{44}^{c}=35/64\ldots$. Thus, it is possible to compute GF (9) as a decomposition of (6) with known and simply computable functions G_n and F_n.

In the work [4] the functions F_n were not considered, and for G_n the formula (1.15) is received resulted with a typo (instead of 1 it is necessary to take 1/2). This formula coincides with (13) (up to the factor 4π determining the GF) and in our case it can be written as:

$$G_n(\mathbf{r}-\mathbf{r}')=\frac{4\pi^2}{i\sqrt{u_+u_-}}\sum_{m=0}^{\infty}(n+2m+1/2)H_{n+2m+1/2}^{(2)}(k_0u_+)I_{n+2m+1/2}(k_0u_-)I_{(n+2m)n}^{c}.\ (15)$$

Let's consider one more representation of G_n and F_n on an example G_n. First take the decomposition of GF

$$G=\sum_{m=0}^{\infty}g_m=\frac{1}{4\pi}\sum_{m=0}^{\infty}(-ik_0)^m R^{m-1/2}.$$

Denoting $\alpha^2 = 4\rho\rho' / \left[(\rho+\rho')^2 + (z-z')^2 \right]$, according to (10) we write the decomposition G_n into a double series with coefficients $(-ik_0)^m G_{nm}$, where the integrals are denoted

$$G_{nm} = \frac{2^{m-2}}{\pi^2 \varepsilon_n} \left(\sqrt{\rho\rho'} / \alpha \right)^{m-1} \int_0^{\pi} \cos(n\theta) \left(1 - \alpha^2 \cos(\theta) \right)^{(m-1)/2} d\theta .$$

They can be represented as

$$G_{nm} = \frac{2^{m-2}}{\pi^2 \varepsilon_n} \left(\sqrt{\rho\rho'} / \alpha \right)^{m-1} \int_{-1}^{1} \frac{T_n(x) \left(1 - \alpha^2 x \right)^{(m-1)/2}}{\sqrt{1-x^2}} dx ,$$

and also calculate through elliptic integrals.

3. IEs transformation. Kernels. Convergence

So, the transformation of the equations can be performed on the basis of the transfer of differentiation operators $\nabla \cdot$ $\nabla\nabla \cdot$ and $\nabla \times$ from the coordinate of the observation point in kernel to the source point with the use of integral theorems, which in the one-dimensional case are reduced to integration in parts. This results to the surface charge density $\varsigma = i\omega^{-1} \nabla \cdot \boldsymbol{\eta}$. It is convenient to transfer these operators to weight functions when using functionals or projection algorithms. Another approach is the allocation of the singular part of the integral operator and its reversal (regularization). In particular, after integration by angle and differentiation by *z*, GF acquires a Cauchy-type singularity [12–14], for which and for the logarithmic singularity there are inversion formulas. Differentiation in (1)–(3) requires the definition of singular integrals in the sense of the principal value, for example, by Cauchy. The neighborhood of the singular point of origin is removed from the kernel, and the integral is understood as the limit when the neighborhood is tightened into a point, thus there is a nonintegral term [34]. This procedure is also regularization, and the IE of the 1st kind it is reduced to the IE of the 2nd kind. In numerical algorithms, a differentiated kernel can be computed directly using *G* or its decomposition (10). To modify the kernel, it is possible to introduce new potentials and perform the replacement of variables and functions. In particular, the representation of a vector field through new functions related to its potential and solenoid parts. In 1938, Gallen proposed an equation with the kernel *G*. It is based on the representation of the vector potential through the current density and *G*, followed by the integration of the wave equation for **A**. in this case, there are integration constants, which are determined from the boundary conditions. A similar approach is used in [4]. In fact, the use of the potential **A** rather than the field **E** means integration over the observation point [35]. Such integration is convenient in functionals when transferring operators to weight functions [34]. An approach based on the use of three-dimensional volume equations for vector potentials is described in [34, 35]. For one-dimensional is it is possible to apply the method of direct integration on the observation point [35, 36]. The singularity of a kernel depends on whether differential or integral operators act on it. The integration reduces singularity. The action of the operators ∇, $\nabla\times$ and $\nabla\otimes\nabla$ on G can be transferred to other functions. This, in particular, it can be the current density or the field. The transfer of the differential operator to the current density gives the divergence $\nabla\cdot\mathbf{J}$, i.e. a formulation with respect to the charge density.

Let us consider other approaches to the transformation of IE (4). Considering its right part as a one-dimensional field, we use Helmholtz's theorem:

$$E_{iz}(z) = -grad_z\left(U(z)\right) + rot_z\left(\mathbf{z}_0 V(z)\right) = -\partial_z U(z). \quad (16)$$

Here $grad_z = \mathbf{z}_0\partial_z$, $rot_z\left(\mathbf{z}_0 V(z)\right) = \mathbf{z}_0\partial_z \times \mathbf{z}_0 V(z) \equiv 0$. Helmholtz's theorem uniquely defines the vector field through its solenoid and potential part, i.e. through its rotor and gradient. A one-dimensional field cannot have a solenoid part, which shows a very approximate view of one-dimensional PE for real VAs modeling. It follows that the entire integral in (4) can be written as a gradient. After integration we get the ratio (6), (7), in which $U(z) = -\partial_z E_{iz}(z)$ can be interpreted as the voltage in the gap of the VA. Thus, obtaining equations for potentials leads to a lower kernel singularity than obtaining equations for fields. The PE is a record of the boundary condition for the electric field (its longitudinal component). In the Hallen equation the boundary conditions are written for the vector potential component. The kernel of this IE is function (11).

4. Model of optically pamped CNT

For structures with a surface current, it is convenient IDE [35, 37] $\mathbf{E}_\tau = \mathbf{E}_{0\tau} + \mathbf{E}_{d\tau}$, in which the total tangent field is the sum of the tangent field $\mathbf{E}_{0\tau}$ of the plane wave and the tangent diffraction field on CNT, at that $\boldsymbol{\eta} = \sigma\mathbf{E}_\tau$ and

$$\boldsymbol{\eta} = \xi\mathbf{E}_{0\tau} - ik_0\xi\int_S\left[G(\mathbf{r}_\tau - \mathbf{r}'_\tau)\boldsymbol{\eta}(\mathbf{r}'_\tau) + k_0^{-2}\nabla G(\mathbf{r}_\tau - \mathbf{r}'_\tau)\nabla'\cdot\boldsymbol{\eta}(\mathbf{r}'_\tau)\right]d^2r', \quad (17)$$

Where $\xi = Z_0\sigma$. To reduce the singularity, the stationary functional is formed by multiplying (19) scalar by $\boldsymbol{\eta}^*$, integrating over the CNT surface and transferring the operator action from G to $\boldsymbol{\eta}^*$. Writing (17) in operator form $\boldsymbol{\eta} = \xi \mathbf{E}_{0\tau} + \hat{K}(\boldsymbol{\eta})$, denoting the integration by a scalar product (,) and using the divergence theorem, we obtain a functional with zero stationary value in the form $\Phi = (\boldsymbol{\eta}, \boldsymbol{\eta} - \xi \mathbf{E}_{0\tau} - \hat{K}(\boldsymbol{\eta}))$ or $\Phi = (\boldsymbol{\eta}, \boldsymbol{\eta}) - \xi(\boldsymbol{\eta}, \mathbf{E}_{0\tau}) - ik_0\xi(\boldsymbol{\eta}, \hat{G}(\boldsymbol{\eta})) + ik_0^{-1}\xi(\nabla\cdot\boldsymbol{\eta}, \hat{G}(\boldsymbol{\eta}))$. Here $\hat{G}$ is an integral operator with kernel G. We used the relation $\boldsymbol{\eta}^*\nabla G = \nabla\cdot(G\boldsymbol{\eta}^*) - G\nabla\boldsymbol{\eta}^*$. Since the CNT surface is closed and has no bounding contours, the integral form $\nabla\cdot(G\boldsymbol{\eta}^*)$ over it is zero. The complex value $P = (\boldsymbol{\eta}, \mathbf{E}_\tau)/2 = \sigma^{-1}(\boldsymbol{\eta}, \mathbf{E}_\tau)/2$ is the power developed on CNT, and when $\mathrm{Re}(\sigma) < 0$ there is an amplification of the incident wave. It is really convenient to estimate it with respect to the scattering cross sections at pumping $\tilde{\Sigma}_s$ and without it Σ_s. The calculations show that the equality $\tilde{P}_\Sigma - P_\Sigma = (\tilde{\boldsymbol{\eta}}, \tilde{\mathbf{E}}_\tau)/2 - (\boldsymbol{\eta}, \mathbf{E}_\tau)/2$ is fulfilled with an accuracy of one percent. Here on the left is the radiation power, the tilde denotes the solution when pumped, and without it the dissipative solution.

The equation (19) was solved for CNT armchair and zigzag with lengths $l = 100$ microns and $a = 20$ nm Fig. 2. The Kubo formula was used for the nonequilibrium conductivity of CNTS [21] under optical pumping. The results show a strong dependence of the amplitude η_z on the chemical potential μ_c. At large CNT radii, their conductivity for both configurations is close to that of graphene [33, 34]. We used the basis functions $\varphi_n(z) = z(l - z)z^{n-1}$, neglecting the dependence on φ, and the representation (19). The solution of IDE with a semi-inverted kernel is also obtained. For Fig. 2 the results of such calculations are given. Both algorithms give the same results. Also we have calculated the strip graphene nanoribbon LVs.

5. Conclusions

We present various forms of the Pocklington, Harrington and Hallen equations, as well as other similar IEs obtained by converting these equations with decreasing the degree of singularity of their nuclei and methods for obtaining them, are considered. One-dimensional singular IEs of electrically thin VAs are also considered. The convergence of the THz wave amplification algorithm is considered on the example of LV as an active CNT. The convergence of the algorithms is determined by the singularity of the kernel, the fulfillment of conditions on the edge and the proximity of the selected first basis function to the solution. The best stability corresponds to the second kind Fredholm IE and determined by the accuracy of the kernel calculation.

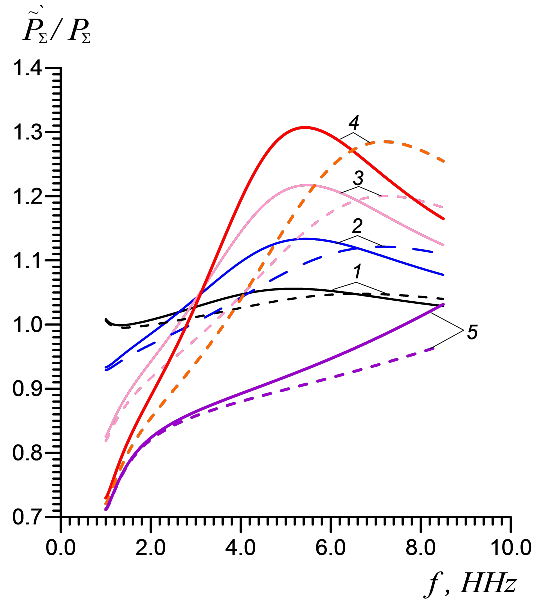

Fig. 2. The ratio of the scattered power of the pumped and conventional CNT at different power densities of the optical pump: 10 (curves 1) , 20 (2), 30 (3) and 40 (4,5) W/cm^2. Solid curves – Armchair (300,300), dashed curves – zigzag (300,120). Curves 1–4 are obtained for l = 10 mm, curves 5 correspond to 1 mm, all CNT parameters correspond to [20].

Рис. 2. Соотношение рассеянной мощности накачиваемой и обычной углеродной нанотрубки при различных плотностях мощности оптической накачки: 10 (кривые 1), 20 (2), 30 (3) и 40 (4,5) Вт/см^2. Сплошные кривые — «кресло» (300,300), пунктирные кривые — зигзаг (300,120). Кривые 1—4 получены для l = 10 мм, кривые 5 соответствуют 1 мм, все параметры углеродной нанотрубки соответствуют [20]

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